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Probabilistic Treatment of Airlift Delivery

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I N S T I T U T E   F O R   D E F E N S E   A N A L Y S E S

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Probabilistic Treatment of Airlift Delivery

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PREFACE

This paper was developed in response to questions regarding how well deterministic transportation models take into account the effects of system reliability. The authors have used a number of transportation models in their career and became interested in how much confidence to place in results that assume perfectly functioning systems when in reality such systems, such as aircraft, may “break” during the operations modeled. The focus chosen here is on a very simple model of air transportation in order to reveal the major issues without the masking complexity of more detailed models. To that end, a simple probabilistic version of a constant rate of delivery model was developed and investigated with statistics. It was supported by the Central Research Program (CRP) at the Institute for Defense Analyses, task C1133. This paper would be of interest to air transportation model developers and users.

The authors appreciate comments from IDA reviewers Dr. Steve Warner (SED Division Director), Dr. Bertrand C. Barrois, Dr. Dennis F. DeRiggi, and Mr. Leo H. Jones. We also appreciate the constructive comments on an earlier briefing of these findings by Col. (USAF) Jean Mahan of the U.S. Transportation Command and Mr. David Merrill of the Air Force Air Mobility Command.

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A. SYNOPSIS

Traditionally, a deterministic simulation is used to estimate airborne cargo and passenger delivery in wartime scenarios. Deterministic models address reliability by removing the number of non-mission capable (NMC) aircraft from the total possessed numbers at the very beginning of the delivery process. Only mission capable (MC) aircraft, minus any special mission withholds, are used in the model. Most important, once an aircraft is deemed to be MC, it never fails anywhere along the delivery and return routes. This way of handling reliability has long been recognized to be a vulnerable feature of such models. In this paper we develop a set of equations that provide quantitative, rapid, first-order observations on how uncertain the results from deterministic model runs can be once stochastic critical part failures are incorporated in the analyses. In particular, we find that the standard deviation of delivery rates, a measure of the uncertainty in the expected results, can be substantial for particularly unreliable air transport aircraft such as the C-5A and even noticeable for the most reliable ones, such as the C-17. The simple results found here should serve as inducement for further research into this important area.

B. BACKGROUND

In recent studies, a deterministic, plane-by-plane simulation determines airborne cargo delivery in wartime scenarios. In the simulation model, such as the AMOS[1] model used at the Air Mobility Command or MIDAS[2] used in the Pentagon by the Office of the Secretary of Defense, the model works as follows. The cargo is assembled according to a time-phased force deployment data (TPFDD) list that identifies which cargo and passengers are available to load and when and at what dates they are required at their destinations. Aircraft are then generated from active, reserve, and USAF Guard units [or from commercial sources if Civil Reserve Air Fleet (CRAF) are used]; they fly to the cargo pickup locations; and cargo is loaded according to what types can be

1 AMOS: Air Mobility Operations Model.

2 MIDAS: Model for Intertheater Deployment by Air and Sea.

accommodated on each specific airlifter type (i.e., all cargo on C-17s and C-5s with preferences for outsized cargo, bulk, and small oversized cargo on C-130s, bulk only on commercial freighters, etc.). The aircraft then flies one of several optional routes, refuels en route either airborne or on the ground, arrives at its off-load destination, and off-loads the cargo. Extra details such as transloading cargo from commercial carriers to military ones at intermediary bases and recovering airlifters at recovery bases before returning to CONUS are also addressed explicitly. One of the main determinants of cargo throughput is also the infrastructure, captured by the values of Maximum-on-Ground (MOG) for individual aircraft at all bases used. The total accumulation of cargo in theater at any point in time is a measure of the delivery capability of a specific fleet. The total accumulated cargo weight divided by the time to that date constitutes the rate of delivery.

C. MODELING RELIABILITY

We wish to focus on the effect of reliability on the rate of delivery. Deterministic models such as AMOS and MIDAS treat reliability in the following fashion. To account for non-reliable aircraft, the NMC aircraft are removed from the Primary Mission Aircraft Inventory (PMAI) aircraft from the beginning. Only MC aircraft are used in the model. Once an aircraft is deemed to be MC, it never fails anywhere along the delivery and return routes. It is assumed to operate with perfect reliability. Non-mission capable aircraft never become mission capable and never take part again in the airlift activities over the time period analyzed. This approach has long been recognized to be a vulnerable feature of such models. It is felt that unreliable aircraft may be accorded too much credit, since the only penalty they incur is that they are used in smaller percentages than more reliable counterparts. What about the aircraft that suffer mission critical part failures en route? Wouldn’t some bases be eliminated from use by other aircraft until the broken aircraft is repaired and takes off? They are not typically modeled in that way.

IDA has embarked on developing a model to explicitly account for mission critical part failures during airlift operations. Our model is the Discrete Airlift Simulation Model (DASM). DASM was initially developed as a part of the 2009 Study on Size and Mix of Airlift Force[3] and is designed to capture the effect of random failures at any time during the cargo delivery process. It can also be used in the same deterministic mode as AMOS or MIDAS for methodology validation.

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[3] *Study on Size and Mix of Airlift Force: Synopsis* (IDA Paper P-4428, UNCLASSIFIED), February 2009.

DASM follows the basic logic of AMOS but with an embedded stochastic failure model used by IDA in past airlifter studies to determine the mission capable rate (MCR) or non-mission capable rate (NMCR)[4] of all airlifter types.[5] Each computer run differs from the previous and succeeding ones through a different random choice of the failure conditions on each run. Hence, the cargo delivered on each day is not represented as a single number of tons delivered, but a probability distribution of weights delivered that day. The answers are probabilistic rather than deterministic. From such distributions, analysts can determine the average weight of cargo delivered as well as other statistics such as uncertainties in the total cargo delivered. Intuitively, the lower the reliability of participating aircraft, the wider will be the distribution and the greater will be the uncertainty.

The DASM model is still in preliminary stages of development and vetting. Can we approximate some of the results we expect to see from such a detailed probabilistic model in a more straightforward way? If so, we can develop simple rules of thumb and quantify our expectations in the absence of detailed simulations. The next section outlines one possible approach.

D. RATE OF DELIVERY AS A RANDOM VARIABLE

When we examine the output from AMOS or MIDAS runs, we see that the rate of delivery is approximately a constant over much of the time, once all airlifters are generated and before nearly all of the air-delivered material has been delivered. So, except for the first few days and the final post-peak delivery days, the cumulative amount of cargo delivered can be considered to be a linear function of time. This is illustrated in Figure 1.

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4 The mission capable rate is the ratio of the number of mission-capable aircraft to the total number of aircraft possessed by the operational command. NMCR is 1-MCR. These are traditionally referred to as “rates,” although “ratios” would be a more accurate term.

5 *Analysis of Alternatives for Out- and Oversize Strategic Airlift: Reliability and Cost Analyses, Volume I: Main Report*, IDA Paper P-3500, March 2000, UNCLASSIFIED.

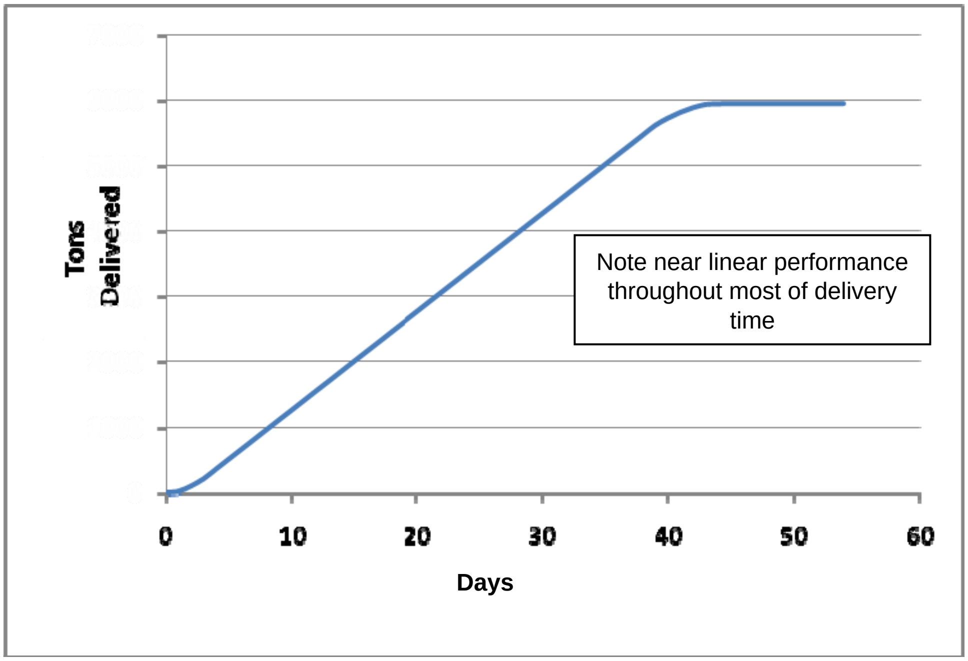

Figure 1. Illustrative Cargo Delivery from Deterministic Simulation Model

The rate of delivery is a constant in this approximation. Call the slope of this straight line R_0, the rate of delivery when only MC airlifters are being used. Note that this assumes that the initially MC aircraft never undergo a reliability failure en route that would slow them down and that the number of such aircraft remains a constant over the period analyzed, i.e., over more than a month if Figure 1 is illustrative.

This is challengeable. We now want to explore the effect of random rates of delivery that result from reliability failures. Let the random rate of delivery be denoted by R, to distinguish it from the constant number R_0. The quantity R is a random variable. Also, in calculating R, we need to use *all* the PMAI aircraft, not just the MC ones. We allow en route delays, to include delay or non-availability at the onset, to account for the MC aircraft not being MC for the entire multi-day delivery process.

To introduce probabilistic behavior, we introduce a probability function $g(t)$, where $g(t)dt$ is the probability of a delay time between t and $t+dt$ as a result of a unavailability, departure delay, or mission-critical parts failure. This probability function embodies all deviations from perfection in a single function, for simplicity. Future treatments can deconvolve this into separate contributions, if that becomes important.

We use the two-sided brackets to denote averages over this distribution. For an arbitrary function $Q(t)$ that depends on delay time t, we define:

$$\langle Q \rangle = \int_0^\infty Q(t)\, g(t)\, dt .$$ Eq. (1)

The particular form for *g(t)* is assumed to follow a simple functional dependence on *t*, falling as *t* increases. It is defined by a single variable τ, which is (shown below) the average "delay" time per airlifter per trip. In this paper we assume that *g(t)* is the continuous distribution function shown in Eq.(2):

$$g(t) = \left(\frac{1}{\tau}\right) \exp(-t/\tau).$$ Eq. (2)

This was chosen to simplify the ensuing mathematics while still retaining the essence of the problem. Other distribution functions could have been selected and may be the subject of future investigations. The distribution function is normalized, i.e., satisfies the requirement that

$$\int_0^\infty g\ (t)\, dt \ = \ 1.$$ Eq. (3)

The parameter τ represents the average "delay" time:

$$\langle t \rangle = \int_0^\infty tg\ (t)\, dt \ = \ \tau .$$ Eq. (4)

It may be useful to note at this point that the delay time parameter τ is an aggregation of all contributions to delay. It includes in-transit delays that result from mission-critical failures en route, but also include all other contributors to "delay." These include unavailability of the aircraft before cargo is loaded, delays in loading and take-off, delays en route to the theater that result from mission critical part failures, the time needed to repair them once the airlifter lands, etc. Rather than explicitly model each of these contributions, we are using a single aggregate delay time τ, with delay modeled as a stochastic event that follows *g(t)*, the probability distribution indicated.

The standard deviation of the distribution of delay times is the same as the average delay time τ. That is, $\tau = \sqrt{\left\langle (t - \langle t \rangle \right\rangle^2}$. Thus, the longer the average delay, the wider the distribution of arrival times and the greater the uncertainty associated with assuming a single deterministic arrival time. This anticipates many of the more detailed findings of this paper.

E. DETERMINATION OF DELAY TIME IN TERMS OF NMCR

We do not model all the contributions that would make up estimating τ . That would be one approach but not the one taken here. Instead we derive what τ would have to be in order for certain commonly assumed relationships to occur. This will give us the explicit connection between τ and NMCR.

Thus, the equation that we use to relate the NMCR and the time parameter in the probability function is

$$R_0 = \langle R \rangle . \qquad \text{Eq. (5)}$$

This equation simply says that the rate of delivery in deterministic models (i.e., R_0 must be the same as the average rate of delivery in probabilistic ones (i.e., $\langle R \rangle$). This is the assumption that justifies using deterministic models in the first place. Circumstances may exist under which this expectation fails, but for this paper we assume that, on average, the deterministic versions of simulation models provide an accurate representation of the expected outcome of a probabilistic analysis. If this were not true, the use of deterministic equations is suspect and the conclusions drawn may be misleading. Treatment of that issue is reserved for subsequent analyses.

To explore the implications of Eq. (5), we need to develop the expressions for rate of delivery a bit further. The rate of delivery is equal to the number of aircraft arriving at their destinations per unit time (per hour, for example) times the average cargo weight carried by each aircraft (short tons) and divided by the average transit time (hours, for example) from the air port of embarkation (APOE) to the air port of debarkation (APOD).

For the deterministic rate,

$$R_0 = fup \frac{N}{T_0}, \qquad \text{Eq. (6)}$$

where f is the fraction of aircraft participating in the delivery, u is the “ute rate” (utilization rate) expressed as the fraction of the day during which flights are conducted, p is the average cargo payload weight per plane, N is the number of mission capable aircraft in the fleet, and T_0 is the uninterrupted transit time from APOE to APOD. Conditions in the infrastructure will determine the specific value of “f”, but we won’t worry about that in this approach since we will be interested in relative values in which

this factor is cancelled out. For unlimited space at airfields, f = 0.47 for strategic lift distances of 7,000 nmi or more, according to AFPAM 10-1403 for Air Mobility Planning Factors.[6] In that document, f is called the "productivity factor" in Table 7. The fractional ute rate u is also provided by the AFPAM 10-1403 in Table 6; it is the rate in that document divided by 24 hours per day to make it a dimensionless fraction. For example, the published ute rate for C-17 is 14.5 hours per day in the surge mode. The fractional ute rate would then be u = 14.5/24 = 0.60. Payloads for all airlifters are in Table 3 of that same document.

Equation (6) represents the deterministic rate of delivery. Every aircraft acts the same, with no variations in this approach. On the other hand, the stochastic rate for the full fleet is

$$R = fup\sum_{k=1}^{N_0} \frac{1}{(T_0+t_k)}, \qquad \text{Eq. (7)}$$

where N_0 is the total number of PMAI airlifters, larger than N, the average number of mission-capable ones. The stochastic delay time t_k represents how much later the k^{th} aircraft will arrive, once availability, delay, and mission-critical reliability issues are incorporated. We treat each aircraft independently, each with its own delay time, but the same probability of delay distribution function $g(t)$.

Upon equating Eq. (6) to the average of Eq.(7), i.e., $R_0 + \langle R \rangle$ and eliminating common factors, we find the following expression. Many common factors such as f, u, and p cancel out, hence the need not to address their values in any detail. Only the times and numbers of aircraft involved differ:

$$\frac{N}{T_0} = N_0 \left\langle \frac{1}{T_0 + t} \right\rangle. \qquad \text{Eq. (8)}$$

Introducing n as the fractional NMCR, where

$$n \equiv \frac{N_0 - N}{N_0}, \qquad \text{Eq. (9)}$$

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[6] Air Force Pamphlet 10-1403, *Air Mobility Planning Factors*, Secretary of the Air Force, 18 December 2003.

we find that

$$(1-n) = \left\langle \frac{1}{1+t/T_0} \right\rangle = \frac{1}{\tau}\int_0^\infty \frac{e^{-t/\tau}}{(1+t/T_0)}dt\,. \qquad \text{Eq. (10)}$$

This can be solved to yield an exact implicit equation:

$$(1-n) = \left(\frac{T_0}{\tau}\right)\left\{-\exp(T_0/\tau)Ei(-T_0/\tau)\right\} \qquad \text{Eq. (11a)}$$

or,

$$(1-n) = (x)\left\{-\exp(x)Ei(-x)\right\}, \qquad \text{Eq. (11b)}$$

with

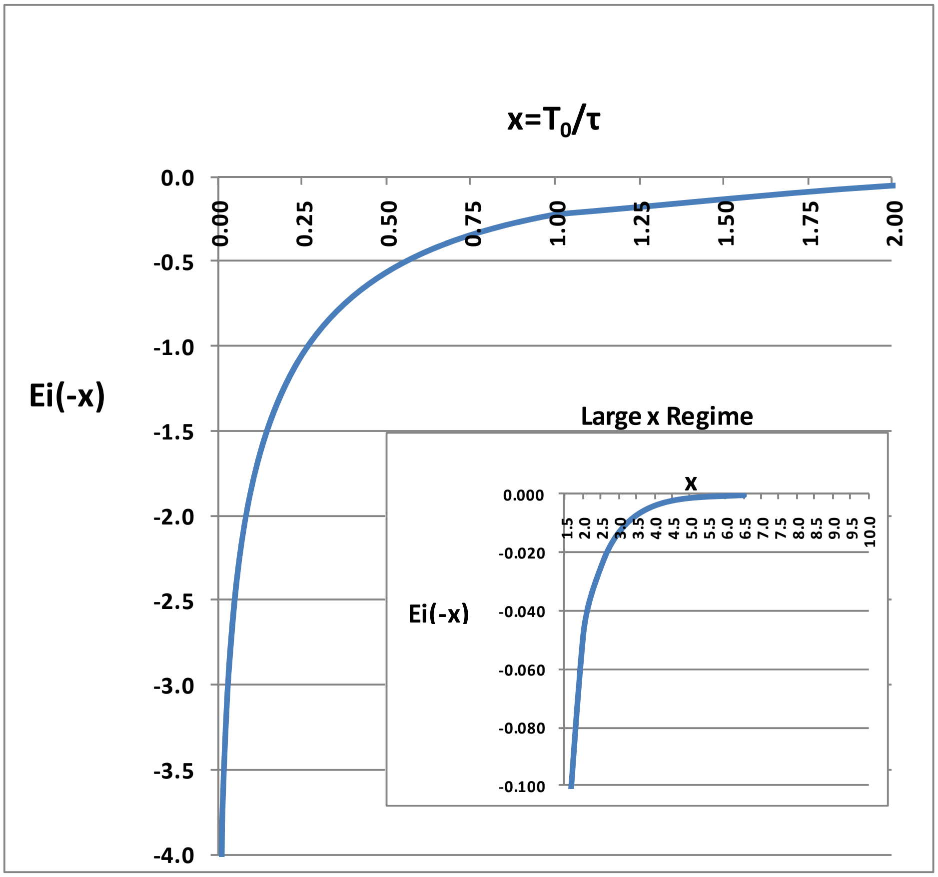
$$x = \left(\frac{T_0}{\tau}\right) \qquad \text{Eq. (11c)}$$

where $Ei(-x)$ is the exponential-integral function. The exponential-integral function[7] is defined to be the integral:

$$Ei(-x) = -\exp(-x)\int_0^\infty \frac{e^{-t}}{t+x}dt\,, \quad \text{for } x>0. \qquad \text{Eq. (12)}$$

This special function in Eq. (12) is negative for all positive values of x and is plotted in Figure 2. The most relevant portion of the x-axis is large x, since this corresponds to small values of the non-mission capable rate. That portion is shown as an insert in Figure 2.

[7] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals and Products*, 4th edition, Academic Press, 1965.



Derived from tables in: Abramowitz and Stegun, *Handbook of Mathematical Functions,1964.*

Figure 2. Exponential-Integral Function Ei(-x)

The expression in Eq. (11) serves as the general implicit relationship that defines τ in terms of *n*, the NMCR value for a class of airlifters. This can be solved iteratively for any value of *n* to find the corresponding values of $(\tau / T_0) = 1/x$. Alternatively, Eq. (11b) can be solved for *n* as a function of *x* and the results inverted. To do either requires a table of exponential-integral functions such as those in the standard mathematical functions handbook by Abramowitz and Stegun.[8]

The results are shown in graphical form in Figure 3. In that graph, the delay time τ relative to the travel time T_0 is shown as a function of the non-mission capable rate *n*. There is high linearity in τ vs. *n* for values of *n* less than about 0.1 but the dependence becomes strongly non-linear as *n* increases beyond 0.3. As *n* approaches unity, the delay time diverges asymptotically to infinity. This divergence occurs because uncertainty

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[8] M. Abramowitz and I.A. Stegun, eds., *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables,* National Bureau of Standards Applied Mathematics Series 55, June 1964.

grows without bound as the aircraft become less reliable. The regime of greatest reality would be values of n less than 0.5.

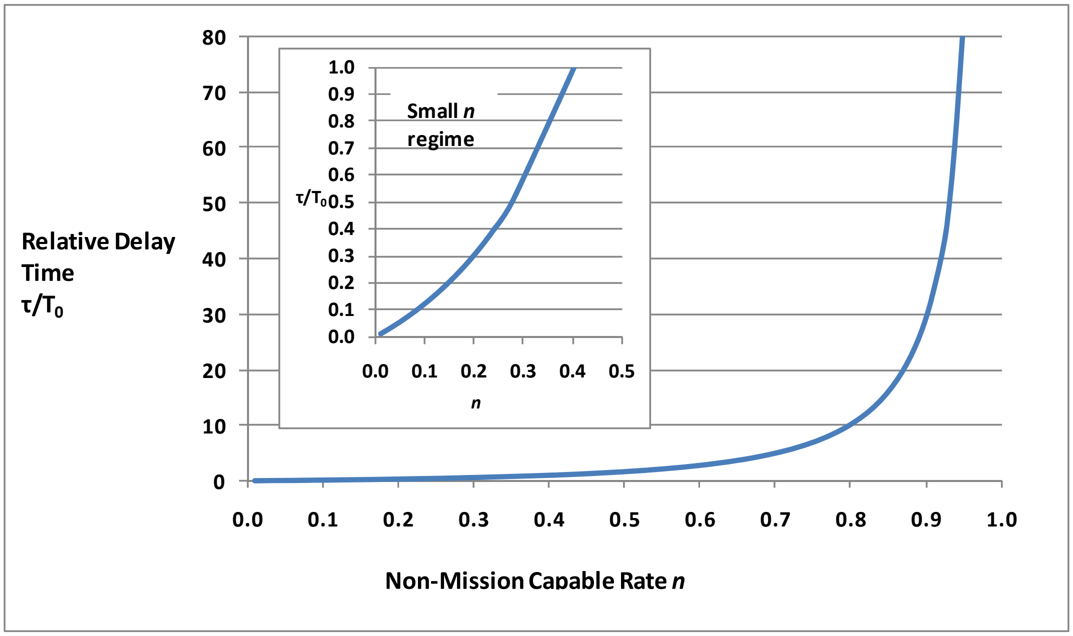

Figure 3. Dependence of the Air Transport Delay Time on Non-Mission Capable Rate

F. UPPER BOUND FOR *x*, LOWER BOUND FOR *τ*

Before proceeding further with the analyses, we next derive bounding results for the expressions in Eq. (11). This is done to possibly provide a simple approximation to Eq. (11) and to guide other attempts to derive approximations.

We use the mathematical property of random variables and statistics that the average of the reciprocal is always larger than the reciprocal of the average. This follows from Jensen's Inequality[9] for positive convex functions such as 1/Z, where Z is a random variable.

Jensen's Inequality says that Eq. (10) can be written as follows

$$(1-n) = \left\langle \frac{1}{1+t/T_0} \right\rangle \geq \frac{1}{1+<t>/T_0} = \frac{1}{1+1/x}. \qquad \text{Eq. (13)}$$

9 E. L. Lehman, *Theory of Point Estimation*, John Wiley & Sons Publishers, New York, 1983, p. 50.

In this derivation, we used Eq. (4) where

$$\langle t \rangle = \int_0^\infty tg(t)dt = \tau = T_0 / x. \qquad \text{Eq. (14)}$$

Solving Eq. (13) for x in terms of the non-mission capable rate n, we find

$$x \le \frac{1-n}{n}. \qquad \text{Eq. (15)}$$

That is, the expression *(1-n)/n* serves as an *upper bound* to the actual value of x for all values of n.

We shall see later this bound is not a good approximation for x. It far exceeds the actual numerical values found from solving Eq. (11) for realistic values of n. While Eq. (15) serves as an upper bound estimate for x, it exaggerates the actual results. This can be easily seen in Figure 4 in which the exact solution from Eq. (11) is contrasted with the bound indicated in Eq. (15). While the solutions appear very close for small values of n, the results quickly diverge as n reaches or exceeds 0.1. Most airlifters have non-mission capable rates ranging out to 0.4 or 0.5, so we need better approximations than the bounding one to represent realistic values of n. For even larger values of n, the differences between the exact solution of Eq. (11) and the bounding conditions expressed by Eq. (15) become more pronounced.

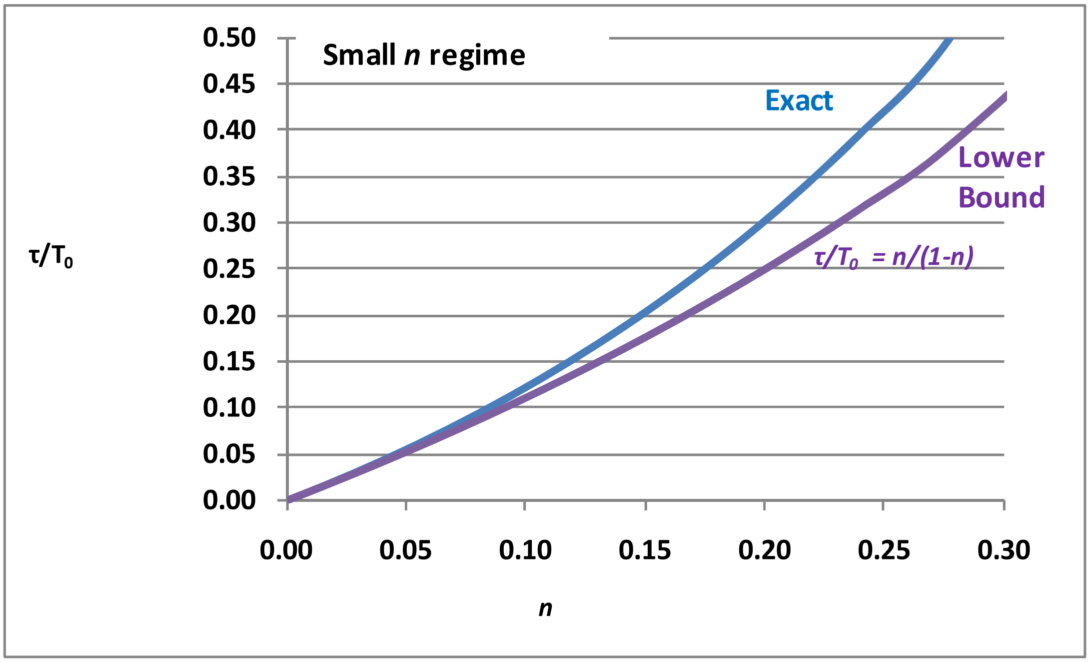

Figure 4. Exact and Lower Bound Values for *τ* vs. *n*

G. APPROXIMATION FOR SMALL NMCR

For very reliable aircraft with *n* near zero (and therefore, large values of *x*), the following asymptotic expansion[10] in *x* is appropriate:

$$Ei(-x) = e^{-x}\sum_{k=1}^{m}\frac{(k-1)!}{(-x)^k} + J_m, \ \ where\ the\ remainder\ |J_m| < \frac{m!\sqrt{2}}{|x|^{m+1}}. \qquad \text{Eq. (16)}$$

This is an interesting area to seek approximations, since it corresponds to the values of *n* and *x* most likely to be encountered. We now use these limiting forms to develop an approximate form for *x* as a function of *n* for all values of *n*.

As seen from Eq. (16), the leading two terms in the expansion of *Ei(-x)* in the small *n*-limit (large *x* limit) is

$$Ei(-x) = e^{-x}\left(-\frac{1}{x} + \frac{1}{x^2}\right). \qquad \text{Eq. (17)}$$

Use of Eq. (11) for the connection between *n* and *x* in this case yields

$$(1-n) = (x)\{-\exp(x)Ei(-x)\} \text{ or}$$

$$(1-n) = 1 - \frac{1}{x},$$

$$x = \frac{1}{n} \qquad \text{Eq. (18)}$$

provided that n remains small. As the NMCR approaches zero (a fleet that does not suffer failures), the delay time that results from unreliability also approaches zero, a reasonable relationship. This looks very familiar from the solution depicted in Figure 3, for small *n*.

However, Eq. (18) is not in accord with the inequality in Eq. (15). This expression for *x* is larger than allowed by Eq. (15), although the error becomes small as *n* approaches zero.

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[10] M. Abramowitz and I.A. Stegun, eds., *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series 55, June 1964.

A better approximation, or at least one that satisfies the bounding relationship better, is to retain one additional term in the asymptotic expansion of the exponential-integral function. In this case, the approximate solution becomes a quadratic equation:

$$(1-n) = 1 - \frac{1}{x} + \frac{2}{x^2}\text{, or} \qquad \text{Eq. (19)}$$

$$\frac{2}{x^2} - \frac{1}{x} + n = 0. \qquad \text{Eq. (20)}$$

This quadratic equation has the solution (of the two $\pm$ solutions, only the negative solution is physically meaningful):

$$\frac{1}{x} = \frac{1-\sqrt{1-8n}}{4}. \qquad \text{Eq. (21)}$$

Figure 5 shows the comparison of the two approximate solutions of *1/x* vs. *n* obtained from the asymptotic expansion of the exponential-integral function with the exact and bounding solutions. Results are shown for n less than 0.3.

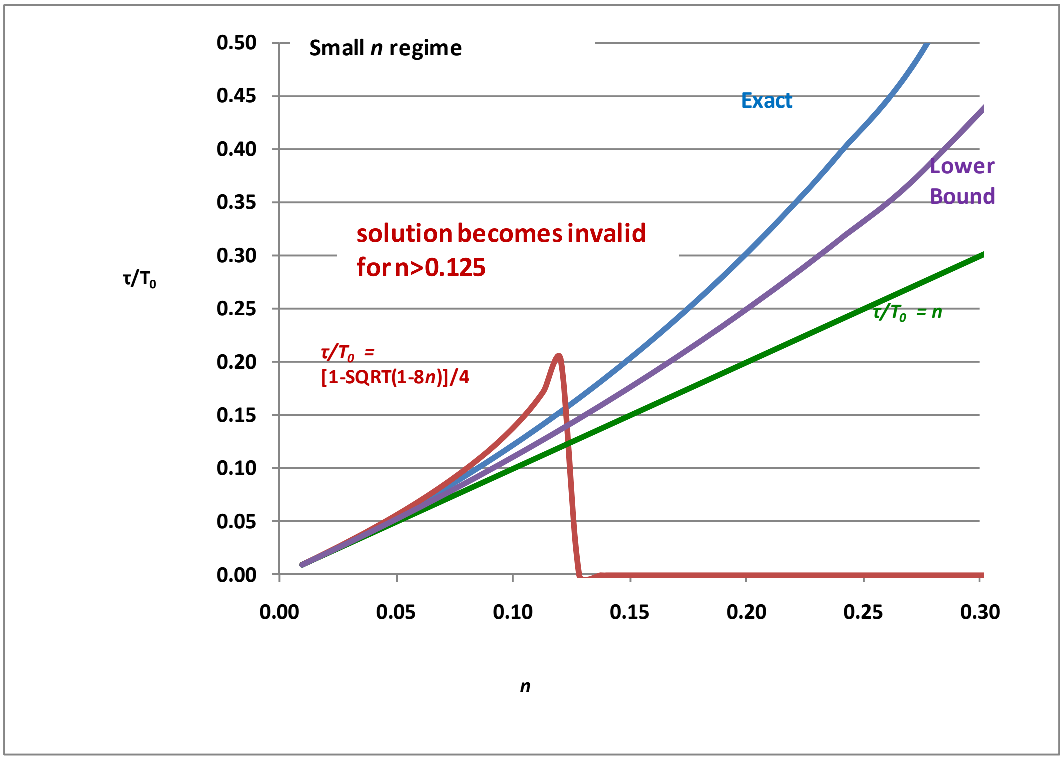

Figure 5. Comparison of Approximate and Exact Values of *τ*

Unfortunately, the expression from Eq. (21) becomes invalid for $n > 1/8 = 0.125$ because it involves the square root of (1-8n), a term that becomes negative for values of n exceeding 1/8.

For very small values of n, i.e., values less than 0.06, Figure 6 compares the exact, bounding, and the two approximations for τ as a function of n. The results can be seen to be all very close for such small values of n. The best fit in this regime is from Eq. (21). Unfortunately, as already noted, that approximation cannot be used for more realistic values of n that exceed $n = 1/8 = 0.125$ so serves as a poor approximation overall.

In summary, the solutions derived from series expansions in the vicinity of large x and small n prove to be poor approximations to the exact solutions for values of n that would be of practical significance. We will return to discussing approximate forms for x as a function of n after addressing the standard deviation in Section H. There we will discover semi-empirically a significantly better relationship, but one not obvious from asymptotic expansions such as were developed in this section. We are still investigating under what assumptions that better fit can be derived rather than inferred.

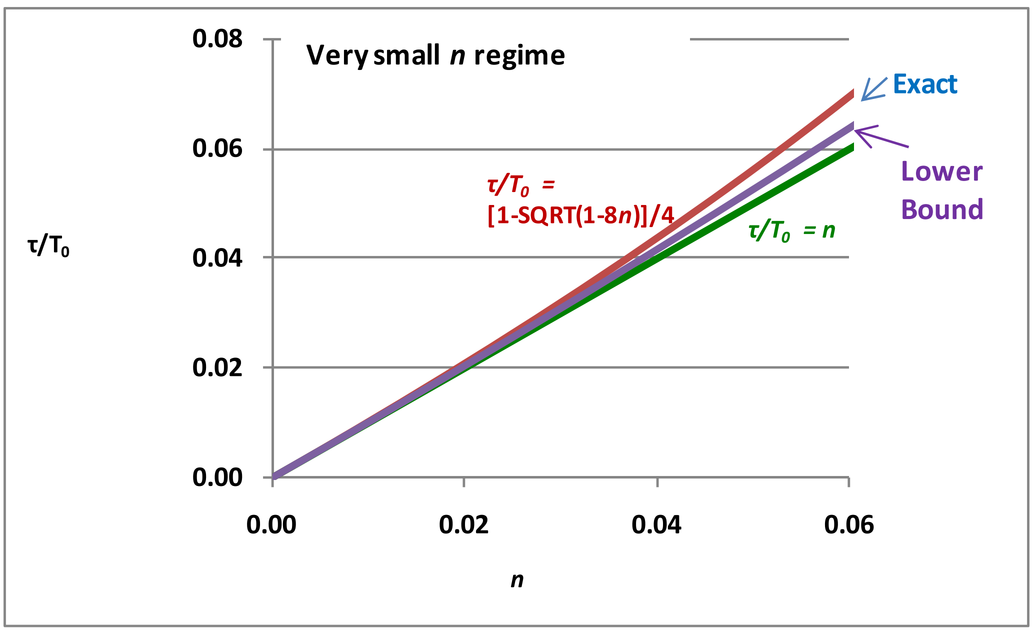

Figure 6. Comparison of Approximate and Exact Values of τ for Very Small n

H. STANDARD DEVIATION OF THE DELIVERY RATE DISTRIBUTION

As noted earlier, the rate of delivery is a random variable, governed by a probability distribution. The average rate was used to relate the average delay time to the NMCR. In this section, we determine the width of the distribution function, as a measure of uncertainty that follows from system unreliability. The expectation is that, the less reliable the airlifter, the more uncertain the outcome of a cargo delivery and the larger the standard deviation, a measure of distribution width. The purpose of this section is to quantify that feature.

The uncertainty is measured by the size of the standard deviation. This in turn is defined as the square root of the distribution variance, where the variance for the rate of delivery distribution function is defined to be the average squared deviation in the delivery rate from the mean:

$$\sigma^2 = \left\langle (R - R_0)^2 \right\rangle \text{, or} \qquad \text{Eq. (22)}$$

$$\sigma^2 = R_0^2 \left\langle \left(\frac{R}{R_0} \right)^2 - 1 \right\rangle . \qquad \text{Eq. (23)}$$

Using expressions for R and R_0 from Eqs. (6) and (7), and cancelling common parameters such as cargo weight, etc, we see that

$$\sigma^2 = R_0^2 \left[\left(\frac{1}{1-n} \right)^2 \left\langle \left(\frac{1}{1+t/T_0} \right)^2 \right\rangle - 1 \right] \qquad \text{Eq. (24)}$$

From Eqs. (11a-c) and the relationship[11]

$$\int_0^\infty \frac{e^{-xt}}{(1+t)^2} = x\{\exp(x) Ei(-x)\} + 1 \qquad \text{Eq. (25)}$$

we find

$$\sigma^2 = R_0^2 \left[\frac{xn}{(1-n)^2} - 1 \right]. \qquad \text{Eq. (26)}$$

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[11] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals and Products*, 4th edition, Academic Press, 1965.

This is an exact solution, with x determined as a function of n by the implicit functional relationships of Eqs. (11a-c) involving exponential-integral functions.

Use of x from Eq. (15) provides an upper bound on the variance. Use of other approximate forms for x provides an approximate result for the variance. When used in the expression for the uncertainty, Eq. (26), we find that

$$UpperBound \quad \sigma^2 \le R_0^2\left[\frac{xn}{(1-n)^2}-1\right]=R_0^2\left[\frac{2n-n^2}{(1-n)^2}\right]. \qquad \text{Eq. (27)}$$

I. GRAPHICAL DEPICTION OF UNCERTAINTY IN DELIVERY RATES

Figure 7 shows a graph of the uncertainty in delivery rate obtained from this analysis. We show the relative uncertainty, defined to be the ratio of the standard deviation of the delivery rate to the average rate. It is given as the square root of the variance in Eq. (26) above. Note the non-linear dependence of the uncertainty, as quantified by the relative standard deviation σ/R_0 , on NMCR. It is linear for small NMCR values near $n = 0$, but deviates as n becomes larger. If the chart were to include values near n = 1, the line would diverge asymptotically to infinity.

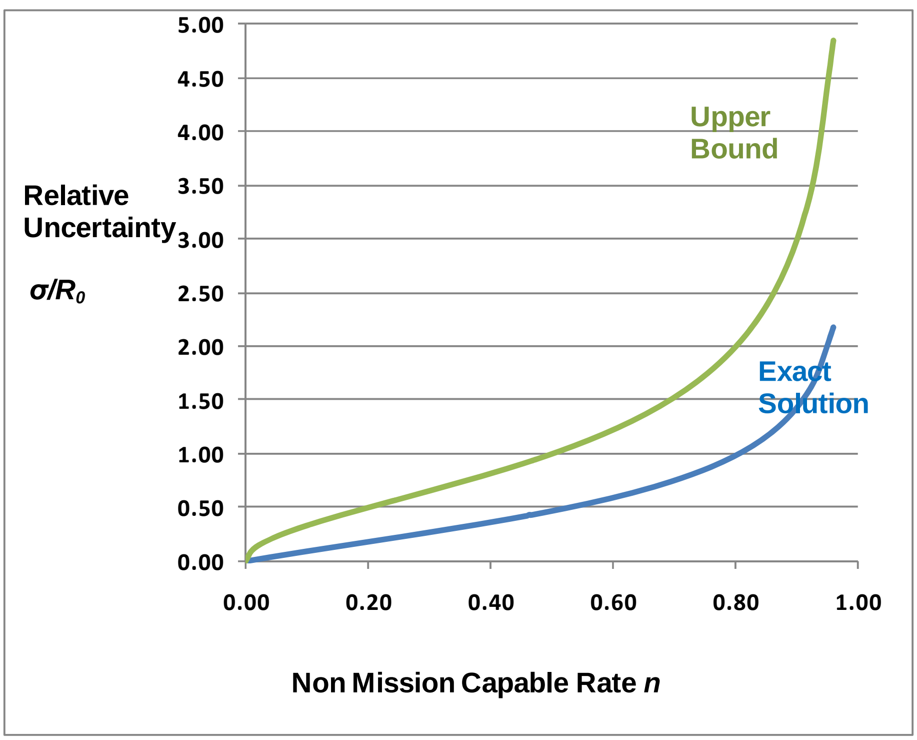

Figure 7. Uncertainty in Rate of Cargo Delivery for All Values of *n*

The results in Figure 7 also show that the upper bound analysis provides a poor approximation to the exact uncertainty. This was already anticipated when discussing the relationship between τ and n. But Figure 7 allows us to speculate on what would be a good approximation, perhaps better than those derived earlier in this paper from series expansions near n = 0. The approximation that the relative uncertainty in delivery rates is the same as the non-mission capable rate seems from observation of the results in Figure 7 to be a reasonable one for small and medium values of *n*. This is a semi-empirical observation, not one derived from asymptotic expansions. Let us pursue the implications of assuming that $\sigma/R_0 = n$.

As Figure 8 shows, this approximation is excellent for small *n* but breaks down badly for large *n*, beyond about 0.70. But this is a vast improvement on approximations derived so far. This deviation for large *n* may be of little practical significance, since aircraft that unreliable are not likely to remain in the fleet. But, from a mathematical point of view as contrasted with a practical one, it does beg for a better approximation across the entire range of *n*-values and one that can be derived rather than induced. Work is underway on that.

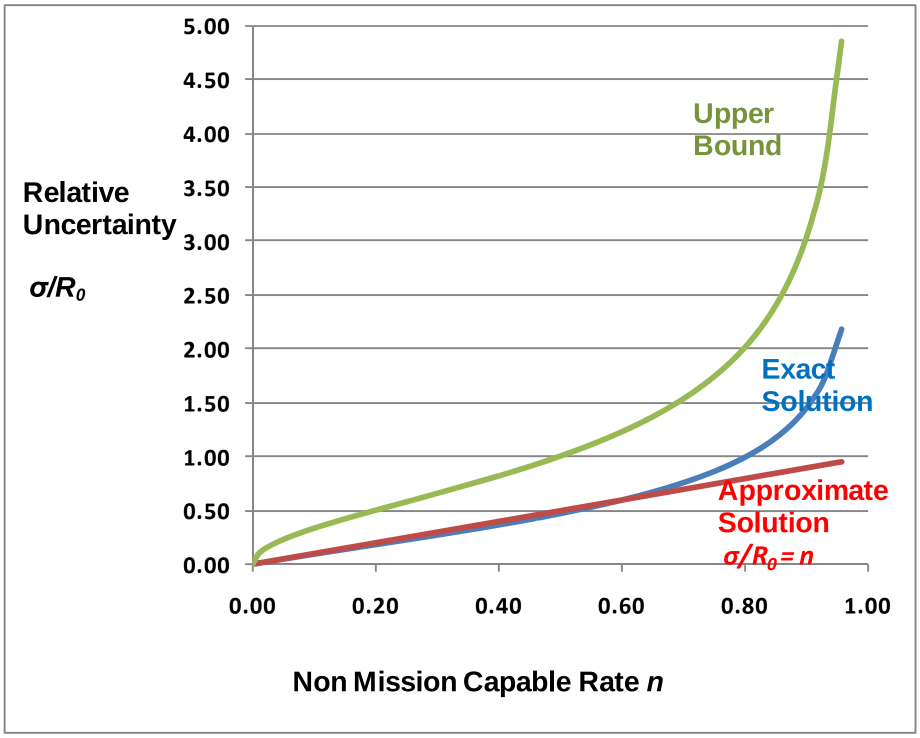

Figure 8. Uncertainty in Rate of Cargo Delivery for All Values of *n*

Focusing on the small *n*-regime, this time for *n* less than 0.5, we find the results depicted in Figure 9. Here the linear approximation for the delivery rate uncertainty is seen to be a very good one, at least in this regime. It slightly exceeds the exact solution, but never by much. By contrast and as anticipated, the upper bound solution proves to be a poor one.

In addition, in Figure 9, we overlay the actual results for selected strategic aircraft: C-17, C-5A, and C-5B. The values of *n* used in this illustration are from the 2000 IDA Study on Over- and Outsize Strategic Airlift.[12]

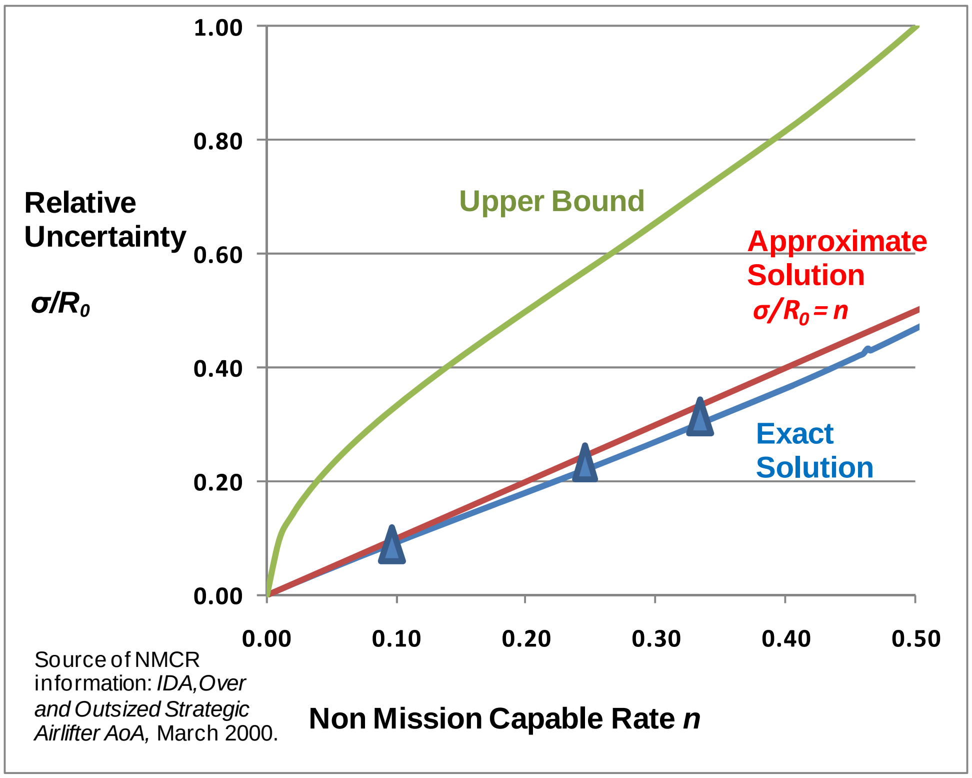

Figure 9. Uncertainty in Rate of Cargo Delivery for Small *n*

Note that for the C-17, with a surge non-mission capable rate of 0.10, there is a relative uncertainty of approximately 0.10. Even for a fleet of 90-percent mission capable airlifters, the delivery rate is a bit uncertain, if this simplified construct provides a good representation of the uncertainty.

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[12] *Analysis of Alternatives for Out- and Oversize Strategic Airlift: Reliability and Cost Analyses, Volume I: Main Report*, IDA Paper P-3500, March 2000, UNCLASSIFIED.

For less reliable airlifters, such as the C-5, the relative uncertainty is even larger. As the figure shows, the relative uncertainty of the C-5B delivery rates is about 0.21 and for the C-5A it is about 0.3. The uncertainty grows without bounds as the non-mission capable rate approaches 1. Figure 9 only shows the more realistic values less than 0.5.

The mission capable rate for the new C-5M is predicted to be better than C-5B but not as good as C-17. If it achieves a non-mission capable rate of 0.2, for example, the uncertainty would be approximately 0.19, according to the figure.

The NMCR values in Figure 9 are for wartime surge conditions under which some waivers are granted. The non-mission capable rate of strategic airlifters is considerably larger in peacetime operations, so the uncertainties would increase accordingly. Since all these values change over time, the reader should use the latest list of non-mission capable rates if making observations about specific airlifters today.

J. IMPLICATIONS OF RELATIVE UNCERTAINTY EQUAL TO NMCR

In the spirit of developing good approximations to the complicated exact solution in Eq. (11), what are the implications of the empirical observation that σ/R_0, the relative uncertainty in the delivery rate, is the same as *n*, the non-mission capable rate?

We can determine that from revisiting Eq.(26) and substituting n^2 for $(\sigma/R_0)^2$.

In other words,

$$\frac{\sigma^2}{R_0^2} = \left[\frac{xn}{(1-n)^2} - 1\right] = n^2. \qquad \text{Eq. (27)}$$

Solving for *x* as a function of *n*, we find that

$$x = \frac{(1+n^2)(1-n)^2}{n}. \qquad \text{Eq. (28)}$$

This has the proper behavior as *n* approaches zero and $\tau/T_0 = 1/x$ approaches *n* from above. This is the same behavior observed in Figure 3. It also has the correct asymptotic behavior as *n* approaches unity, although we know from analyses earlier in Figure 8 that it is not a good approximation for *n* much larger than 0.7.

What are the implications of this approximation on the dependence of τ on *n*? Figure 10 shows this relationship. It is an outstanding approximation even out to n = 0.5

and beyond. While not obvious from basic asymptotic expansion assumptions, this semi-empirical relationship is clearly an excellent one.

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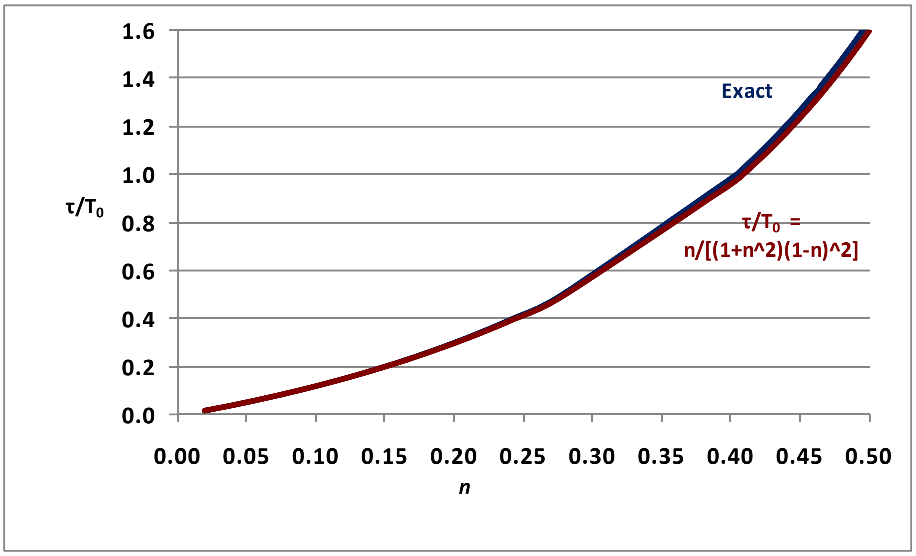

Figure 10. Comparison of Exact and Approximate Calculations of τ vs. n, for n < 0.5

Figure 11 shows the behavior of the approximation in Eq. (28) for all n, just to illustrate where it is a good approximation and where it begins to fail. That is it poor for n greater than 0.7 is probably not of practical interest.

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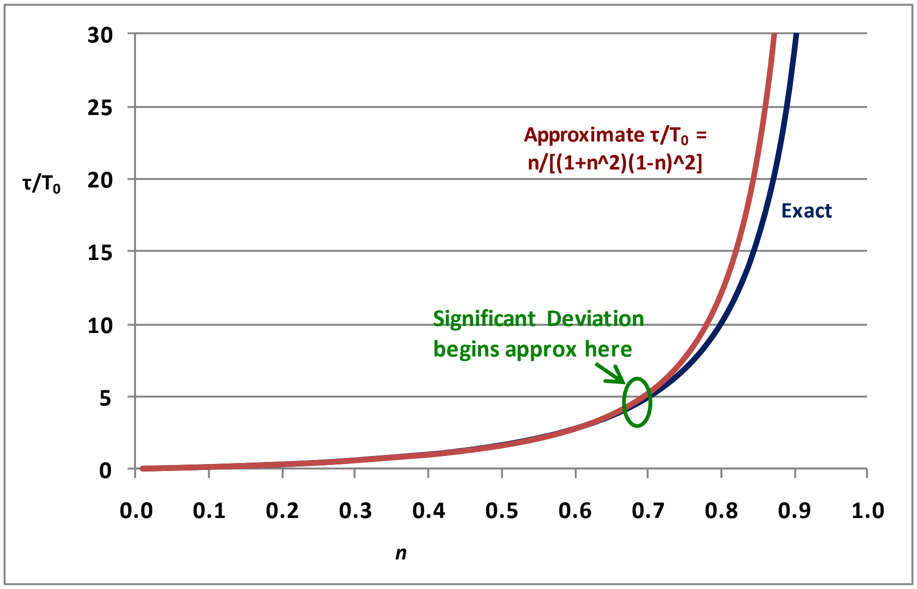

Figure 11. Comparison of Exact and Approximate Calculations of τ vs. n, for All n

K. SUMMARY

We have derived exact, bounding, and approximate solutions for the delay and uncertainty associated with unreliable airlifters involved in massive and continuous strategic lift. Since the exact solutions involve less commonly encountered functions, namely the exponential-integral function, approximate solutions that aid in visualization were sought. We identified one simple approximation and are exploring additional ones that may be more appropriate over a larger set of values for the NMCR.

Central to the theory is the non-mission capable rate, an operational factor. Non-mission capable rate is represented in our treatment by the parameter n, a positive fraction between 0 and 1. The larger the value of n, the more unreliable the aircraft.

1. Exact Solutions

The parameters that we derive are

Exact Delay $$(1-n) = -x\exp(x)Ei(-x),$$ Eq. (29)

Exact Uncertainty $$\left(\frac{\sigma}{R_0}\right) = \sqrt{\frac{xn}{(1-n)^2} - 1}.$$ Eq. (30)

Eq. (29) serves as an implicit equation for the parameter x in terms of the non-mission capable rate n. Once x is determined for a specific value of n, it is used in Eq. (30) for the relative standard deviation, a measure of delivery uncertainty. Graphical illustrations of both solutions are displayed in this paper.

While exact solutions can be obtained from a look-up table for the exponential-integral function, it is handy to have approximate but very good relationships that do not require such exotic functions. These we have also derived.

2. Approximate Solutions

We also simplified the formal expression and derived useful and fairly accurate approximate solutions:

Approx Delay $$\frac{1}{x} = \frac{\tau}{T_0} \approx \frac{n}{(1+n^2)(1-n)^2},$$ Eq. (31)

Approx Uncertainty $$\left(\frac{\sigma}{R_0}\right) \approx n.$$ Eq. (32)

These approximate solutions in Eqs. (31) and (32) are most appropriate when the NMCR is smaller than 0.7, a condition usually found for operational units. Even the worst strategic airlifters have *n*-values less than 0.5 and well within the region for which these expressions provide good approximations to the exact solutions. They are also consistent with the bounding solutions, shown below.

3. Bounding Solutions

The upper or lower bounds for the parameters are

Delay Bound $$\frac{1}{x} = \frac{\tau}{T_0} \geq \frac{n}{(1-n)},$$ Eq. (33)

Uncertainty Bound $$\left(\frac{\sigma}{R_0}\right) \leq \sqrt{\frac{n}{(1-n)}}\,.$$ Eq. (34)

In general, these themselves provide a poor approximation to the exact solutions, for which the approximate solutions in Eqs. (31) and (32) are better.

4. Implications

The significant standard deviations derived for airlift aircraft, especially aircraft of high non-mission capable rates, raise concern about using deterministic models to characterize airlift and to compare delivery rates for different fleets. Fleet size and mix analyses of alternatives have used these models for several decades without questioning how reliable they are, especially in the relative comparisons of fleets that differ substantially in aircraft reliability.

Our analyses here—with an admittedly simple model—suggest that probabilistic models should be pursued more intensely in this field. The next step would be to compare actual probabilistic models such as DASM against the predictions derived here and then extend them to circumstances more realistically characterizing the transportation infrastructure for theaters of interest.

Appendix A
GLOSSARY

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GLOSSARY

AFPAM	Air Force Pamphlet
AMOS	Air Mobility Operations Model
APOD	air port of debarkation
APOE	air port of embarkation
CONUS	continental United States
CRAF	Civil Reserve Air Flee
CRP	Central Research Program
DASM	Discrete Airlift Simulation Model
MC	mission capable
MCR	mission capable rate
MIDAS	Model for Intratheater Deployment by Air and Sea
MOG	Maximum-on-Ground
NMC	non-mission capable
NMCR	non-mission capable rate
PMAI	Primary Mission Aircraft Inventory
TPFDD	time-phased force deployment data
USAF	United States Air Force
ute	utilization

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14. ABSTRACT

Traditionally, a deterministic simulation is used to estimate airborne cargo and passenger delivery in wartime scenarios. Deterministic models address reliability by removing the number of non-mission capable aircraft from the total possessed numbers at the very beginning of the delivery process. Only mission capable (MC) aircraft, minus any special mission withholds, are used in the model. Most important, once an aircraft is deemed to be MC, it never fails anywhere along the delivery and return routes in the deterministic models. This has long been recognized to be a vulnerable feature of such models. How inaccurate is this? What problems can arise from its assumption? In this paper we present a set of equations that provide quantitative, rapid, but simple first-order observations on how uncertain the results from deterministic model runs can be, once stochastic critical part failures are incorporated in the analyses. In particular, we find that the standard deviation of delivery rates, a measure of the uncertainty in the expected results, can be substantial for particularly unreliable air transport aircraft such as the C-5A and even noticeable for the most reliable ones, such as the C-17. The simple results found here should serve as an incentive for further research into this important area.

15. SUBJECT TERMS

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